On q-orthogonal polynomials, dual to little and big q-Jacobi polynomials

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Abstract

This paper studies properties of q-Jacobi polynomials and their duals by means of operators of the discrete series representations for the quantum algebra $U_q(\mathrm{su}_{1,1})$. Spectra and eigenfunctions of these operators are found explicitly. These eigenfunctions, when normalized, form an orthonormal basis in the representation space. The initial $U_q(\mathrm{su}_{1,1})$-basis and the bases of these eigenfunctions are interconnected by matrices, whose entries are expressed in terms of little or big q-Jacobi polynomials. The orthogonality by rows in these unitary connection matrices leads to the orthogonality relations for little and big q-Jacobi polynomials. The orthogonality by columns in the connection matrices leads to an explicit form of orthogonality relations on the countable set of points for $_3\phi_2$ and $_3\phi_1$ polynomials, which are dual to big and little q-Jacobi polynomials, respectively. The orthogonality measure for the dual little q-Jacobi polynomials proves to be extremal, whereas the measure for the dual big q-Jacobi polynomials is not extremal.

Key words. Orthogonal q-polynomials, little q-Jacobi polynomials, big q-Jacobi polynomials, Leonard pairs, orthogonality relations, quantum algebra

AMS subject classification. 33D80, 33D45, 17B37

1. Introduction

The appearance of quantum groups and quantized universal enveloping algebras (quantum algebras) and development of their representations led to their applications in the theory of q-orthogonal polynomials and q-special functions (see, for example, [1–4]). Since the theory of quantum groups and their representations is much more complicated than the Lie theory, the corresponding applications are more difficult. At the first stage of such applications, the compact quantum groups and their finite dimensional representations have been used.

It is known that representations of the noncompact Lie group $SU(1,1) \sim SL(2,\mathbb{R})$ are very productive for the theory of orthogonal polynomials and special functions (see, for example, [5], Chapter 7). Unfortunately, there are difficulties with a satisfactory definition of the noncompact quantum group $SU_q(1,1)$, which would give us a possibility to use such quantum group extensively for deeper understanding the theory of orthogonal polynomials and special functions. For this reason, representations of the corresponding quantum algebra $U_q(\mathrm{su}_{1,1})$ have been commonly used for such purposes (see, for example, [6–12]).

In this paper we continue to use representations of the positive discrete series of the quantum algebra $U_q(\mathrm{su}_{1,1})$ for exploring properties of q-orthogonal polynomials [6–8]. In fact, we deal with certain operators in these representations and do not touch the Hopf structure of the algebra $U_q(\mathrm{su}_{1,1})$. Our study of these polynomials is related to representation operators, which can be represented by a Jacobi matrix. Namely, we consider those representation operators A, which correspond to some particular Jacobi matrices. In the case under discussion we

diagonalize these selfadjoint bounded operators with the aid of big or little q-Jacobi polynomials. An explicit form of all eigenfunctions of these operators is found. Since the spectra are simple, eigenfunctions of each such operator form an orthogonal basis in the representation space. One can normalize this basis. This normalization is effected by means of the second operator J, which is related (in some sense) to q-difference equations for little and big q-Jacobi polynomials. As a result of the normalization, for each operator A (one of them is related to little q-Jacobi polynomials and another one to big q-Jacobi polynomials) two orthonormal bases in the representation space emerge: the canonical (or the initial) basis and the basis of eigenfunctions of the operator A. They are interrelated by a unitary matrix U whose entries u_{mn} are explicitly expressed in terms of little or big q-Jacobi polynomials. Since the matrix U is unitary (and in fact it is real in our case), there are two orthogonality relations for its elements, namely

$$\sum_n u_{mn} u_{m'n} = \delta_{mm'}, \quad \sum_m u_{mn} u_{mn'} = \delta_{nn'}. \tag{1.1}$$

The first relation expresses the orthogonality relation for little or big q-Jacobi polynomials. So, the orthogonality of U yields an algebraic proof of orthogonality relations for these polynomials. In order to interpret the second relation, we consider little and big q-Jacobi polynomials $P_n(q^{-m})$ as functions of n. In this way one obtains two sets of orthogonal functions (one for little q-Jacobi polynomials and another for big q-Jacobi polynomials), which are expressed in terms of q-orthogonal polynomials (which can be considered as a dual sets of polynomials with respect to little and big q-Jacobi polynomials; such duality is well known in the case of polynomials, orthogonal on a finite set of points). The second relation in (1.1) leads to the orthogonality relations for these q-orthogonal polynomials on nonuniform lattices.

In fact, this idea extends the notion of the duality of polynomials, orthogonal on a finite set, to the case of polynomials, orthogonal on an infinite set of points. We have already used this idea in [6] and [8] to show that Al-Salam–Carlitz II polynomials are dual with respect to little q-Laguerre polynomials and q-Meixner polynomials are dual to big q-Laguerre polynomials. We emphasize at this point that there are known theorems on dual orthogonality properties of q-polynomials, whose weight functions are supported on a discrete set of points (see, for example, [13] and [14]). However, they are formulated in terms of orthogonal functions (see (5.1) and (8.1) below for their explicit forms in the case of little and big q-Jacobi polynomials, respectively) as dual objects with respect to given orthogonal polynomials. Therefore, one still needs to make one step further in order to single out an appropriate family of dual polynomials from these functions. So, our main motivation for this paper is to show explicitly how to accomplish that for little and big q-Jacobi polynomials.

The orthogonality measure for polynomials, dual to little q-Jacobi polynomials, is extremal, that is, these polynomials form a complete set in the space L^2 with respect to their orthogonality measure.

The orthogonality measure for polynomials, dual to big q-Jacobi polynomials, is not extremal: these polynomials do not form a complete set in the corresponding space L^2. We have found the complementary set of orthogonal functions in this space L^2. These functions are expressed in terms of the same polynomials but with different values of parameters.

The dual little q-Jacobi polynomials and the dual big q-Jacobi polynomials, as other well-known q-orthogonal polynomials, can be applied in different branches of science. For example, they may be useful for studying a certain type of q-difference equations, which appear in applications in engineering and physics.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of special functions and the

standard q-analysis (see, for example, [15] and [16]). In particular, we adopt for q-numbers $[a]_q$ the form

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}, \tag{1.2}$$

where a is any complex number and $q^{1/2}$ is assumed to be positive. We shall also use the well-known shorthand $(a_1, \cdots, a_k; q)_n := (a_1; q)_n \cdots (a_k; q)_n$.

2. Discrete series representations of $U_q(\mathrm{su}_{1,1})$

The quantum algebra $U_q(\mathrm{su}_{1,1})$ is defined as the associative algebra, generated by the elements J_+, J_-, q^{J_0} and q^{-J_0}, subject to the commutation relations

$$q^{J_0} q^{-J_0} = q^{-J_0} q^{J_0} = 1, \quad q^{J_0} J_\pm q^{-J_0} = q^{\pm 1} J_\pm, \quad [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}},$$

and the involution relations $(q^{J_0})^* = q^{J_0}$ and $J_+^* = J_-$. (Observe that here we have replaced J_- by $-J_-$ in the common definition of the algebra $U_q(\mathrm{sl}_2)$.) For brevity, in what follows we denote the algebra $U_q(\mathrm{su}_{1,1})$ by $\mathrm{su}_q(1,1)$.

If the algebra $\mathrm{su}_q(1,1)$ is realized in terms of the operators, one may consider also the operator J_0. In this case instead of the commutation relations between J_+, J_-, q^{J_0} and q^{-J_0}, it is convenient to work with more familiar relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}}.$$

Then the involution relations reduce to the following ones:

$$J_0^* = J_0, \quad J_+^* = J_-. \tag{2.1}$$

We are interested in the discrete series representations of $\mathrm{su}_q(1,1)$ with lowest weights. These irreducible representations will be denoted by T_l^+, where l is a lowest weight, which can be any positive number (see, for example, [17]).

The representation T_l^+ can be realized on the space $\mathcal{L}_l$ of all polynomials in x. We choose a basis for this space, consisting of the monomials

$$f_n^l(x) := c_n^l \, x^n, \quad n = 0, 1, 2, \cdots, \tag{2.2}$$

where

$$c_0^l = 1, \quad c_n^l = \prod_{k=1}^n \frac{[2l + k - 1]_q^{1/2}}{[k]_q^{1/2}} = q^{(1-2l)n/4} \frac{(q^{2l}; q)_n^{1/2}}{(q; q)_n^{1/2}}, \quad n = 1, 2, 3, \cdots, \tag{2.3}$$

and $(a; q)_n = (1 - a)(1 - aq) \ldots (1 - aq^{n-1})$. The representation T_l^+ is then realized by the operators

$$J_0 = x \frac{d}{dx} + l, \quad J_\pm = x^{\pm 1} [J_0(x) \pm l]_q.$$

As a result of this realization, we have

$$J_+ f_n^l = \sqrt{[2l + n]_q [n + 1]_q} \, f_{n+1}^l = \frac{q^{-(n+l-1/2)/2}}{1 - q} \sqrt{(1 - q^{n+1})(1 - q^{2l+n})} f_{n+1}^l, \tag{2.4}$$

3

$$J_- f_n^l = \sqrt{[2l + n - 1]_q [n]_q} f_{n-1}^l = \frac{q^{-(n+l-3/2)/2}}{1 - q} \sqrt{(1 - q^n)(1 - q^{2l+n-1})} f_{n-1}^l, \qquad (2.5)$$

$$J_0 f_n^l = (l + n) f_n^l. \qquad (2.6)$$

We know that the discrete series representations T_l can be realized on a Hilbert space, on which the adjointness relations (2.1) are satisfied. In order to obtain such a Hilbert space, we assume that the monomials $f_n^l(x)$, $n = 0, 1, 2, \cdots$, constitute an orthonormal basis for this Hilbert space. This introduces a scalar product $\langle \cdot, \cdot \rangle$ into the space $\mathcal{L}_l$. Then we close this space with respect to this scalar product and obtain the Hilbert space, which will be denoted by $\mathcal{H}_l$.

The Hilbert space $\mathcal{H}_l$ consists of functions (series)

$$f(x) = \sum_{n=0}^{\infty} b_n f_n^l(x) = \sum_{n=0}^{\infty} b_n c_n^l x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = b_n c_n^l$. Since $\langle f_m^l, f_n^l \rangle = \delta_{mn}$ by definition, for $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $f'(x) = \sum_{n=0}^{\infty} a'_n x^n$ we have $\langle f, f' \rangle = \sum_{n=0}^{\infty} a_n \overline{a'_n} / |c_n^l|^2$, that is, the Hilbert space $\mathcal{H}_l$ consists of analytical functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$, such that $\|f\|^2 \equiv \sum_{n=0}^{\infty} |a_n / c_n^l|^2 < \infty$.

3. Leonard pair (I_1, J)

Let I_1 be the operator

$$I_1 := -\alpha q^{J_0/4}(J_+ A + A J_-)q^{J_0/4} + B \qquad (3.1)$$

of the representation T_l^+, where $\alpha = (a/q)^{1/2} (1 - q)$, $a = q^{2l-1}$, and A and B are operators of the form

$$A = \frac{q^{J_0-l+1/2}\sqrt{(1 - bq^{J_0-l+1})(1 - abq^{J_0-l+1})}}{(1 - abq^{2J_0-2l+2})\sqrt{(1 - abq^{2J_0-2l+1})(1 - abq^{2J_0-2l+3})}},$$

$$B = \frac{q^{J_0-l}}{1 - abq^{2J_0-2l+1}} \left(\frac{(1 - aq^{J_0-l+1})(1 - abq^{J_0-l+1})}{1 - abq^{2J_0-2l+2}} + a\frac{(1 - q^{J_0-l})(1 - bq^{J_0-l})}{1 - abq^{2J_0-2l}} \right).$$

Since the bounded operator q^{J_0} is diagonal in the basis $\{f_n^l\}$, the operators A and B are well defined.

We have the following expression for the action of the operator I_1 in the canonical basis f_n^l, $n = 0, 1, 2, \cdots$:

$$I_1 f_n^l = -a_n f_{n+1}^l - a_{n-1} f_{n-1}^l + b_n f_n^l, \qquad (3.2)$$

where

$$a_n = a^{1/2} q^{n+1/2} \frac{\sqrt{(1 - q^{n+1})(1 - aq^{n+1})(1 - bq^{n+1})(1 - abq^{n+1})}}{(1 - abq^{2n+2})\sqrt{(1 - abq^{2n+1})(1 - abq^{2n+3})}},$$

$$b_n = \frac{q^n}{1 - abq^{2n+1}} \left(\frac{(1 - aq^{n+1})(1 - abq^{n+1})}{1 - abq^{2n+2}} + a\frac{(1 - q^n)(1 - bq^n)}{1 - abq^{2n}} \right).$$

In order to assure that expressions for a_n and b_n are well defined, we suppose that $b < q^{-1}$. Note that $0 < a = q^{2l-1} < q^{-1}$ and l takes any positive value. Since the $q^{\pm J_0}$ are symmetric operators, the operator I_1 is also symmetric.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator I_1 is bounded. Therefore, we assume that it is defined on the whole representation space $\mathcal{H}_l$. For this reason, I_1 is a selfadjoint

4

operator. Let us show that I_1 is a trace class operator (we remind that a bounded selfadjoint operator is a trace class operator if a sum of its matrix elements in an orthonormal basis is finite; a spectrum of such an operator is discrete, with a single accumulation point at 0). For the coefficients a_n and b_n from (3.2), we have

$$a_{n+1}/a_n \to q, \quad b_{n+1}/b_n \to q \quad \text{when} \quad n \to \infty.$$

Therefore, for the sum of all matrix elements of the operator I_1 in the canonical basis we have $\sum_n (2a_n + b_n) < \infty$. This means that I_1 is a trace class operator. Thus, a spectrum of I_1 is discrete and have a single accumulation point at 0. Moreover, a spectrum of I_1 is simple, since I_1 is representable by a Jacobi matrix with $a_n \neq 0$ (see [18], Chapter VII).

To find eigenfunctions $\xi_\lambda(x)$ of the operator I_1, $I_1\xi_\lambda(x) = \lambda\xi_\lambda(x)$, we set

$$\xi_\lambda(x) = \sum_{n=0}^{\infty} \beta_n(\lambda) f_n^l(x).$$

Acting by the operator I_1 upon both sides of this relation, one derives that

$$\sum_{n=0}^{\infty} \beta_n(\lambda) \left(a_n f_{n+1}^l + a_{n-1} f_{n-1}^l - b_n f_n^l\right) = -\lambda \sum_{n=0}^{\infty} \beta_n(\lambda) f_n^l,$$

where a_n and b_n are the same as in (3.2). Collecting in this identity all factors, which multiply f_n^l with fixed n, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)a_n + \beta_{n-1}(\lambda)a_{n-1} - \beta_n(\lambda)b_n = -\lambda\beta_n(\lambda).$$

Making the substitution

$$\beta_n(\lambda) = \left(\frac{(abq, aq; q)_n (1 - abq^{2n+1})}{(bq, q; q)_n (1 - abq)(aq)^n}\right)^{1/2} \beta_n'(\lambda)$$

reduces this relation to the following one

$$A_n\beta_{n+1}'(\lambda) + C_n\beta_{n-1}'(\lambda) - (A_n + C_n)\beta_n'(\lambda) = -\lambda\beta_n'(\lambda)$$

with
$$A_n = \frac{q^n(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \quad C_n = \frac{aq^n(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

This is the recurrence relation for the little q-Jacobi polynomials

$$p_n(\lambda; a, b|q) := {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, q\lambda) \tag{3.3}$$

(see, for example, formula (7.3.1) in [15]). Therefore, $\beta_n'(\lambda) = p_n(\lambda; a, b|q)$ and

$$\beta_n(\lambda) = \left(\frac{(abq, aq; q)_n (1 - abq^{2n+1})}{(bq, q; q)_n (1 - abq)(aq)^n}\right)^{1/2} p_n(\lambda; a, b|q). \tag{3.4}$$

For the eigenfunctions $\xi_\lambda(x)$ we have the expression

$$\xi_\lambda(x) = \sum_{n=0}^{\infty} \left(\frac{(abq, aq; q)_n (1 - abq^{2n+1})}{(bq, q; q)_n (1 - abq)(aq)^n}\right)^{1/2} p_n(\lambda; a, b|q) f_n^l(x)$$

5

$$= \sum_{n=0}^{\infty} a^{-n/4} \frac{(aq;q)_n}{(q;q)_n} \left(\frac{(abq;q)_n (1 - abq^{2n+1})}{(bq;q)_n (1 - abq)(aq)^n} \right)^{1/2} p_n(\lambda; a, b|q) x^n. \tag{3.5}$$

Since the spectrum of the operator I_1 is discrete, only a discrete set of these functions belongs to the Hilbert space $\mathcal{H}_l$. This discrete set of functions determines a spectrum of I_1.

Now we look for a spectrum of the operator I_1 and for a set of polynomials, dual to the little q-Jacobi polynomials. To this end we use the action of the operator

$$J := q^{-J_0 + l} + ab \, q^{J_0 - l + 1}$$

upon the eigenfunctions $\xi_\lambda(x)$, which belong to the Hilbert space $\mathcal{H}_l$. In order to find how this operator acts upon these functions, one can use the q-difference equation

$$(q^{-n} + abq^{n+1}) \, p_n(\lambda) = a\lambda^{-1}(bq\lambda - 1) \, p_n(q\lambda) + \lambda^{-1}(1 + a) \, p_n(\lambda) + \lambda^{-1}(\lambda - 1) \, p_n(q^{-1}\lambda) \tag{3.6}$$

for the little q-Jacobi polynomials $p_n(\lambda) \equiv p_n(\lambda; a, b|q)$ (see, for example, formula (3.12.5) in [19]). Multiply both sides of (3.6) by $d_n \, f_n^l(x)$ and sum up over n, where d_n are the coefficients of $p_n(\lambda; a, b|q)$ in the expression (3.4) for the coefficients $\beta_n(\lambda)$. Taking into account the first part of formula (3.5) and the fact that $Jf_n^l(x) = (q^{-n} + abq^{n+1})f_n^l(x)$, one obtains the relation

$$J \, \xi_\lambda(x) = a\lambda^{-1}(bq\lambda - 1) \, \xi_{q\lambda}(x) + \lambda^{-1}(1 + a) \, \xi_\lambda(x) + \lambda^{-1}(\lambda - 1) \, \xi_{q^{-1}\lambda}(x). \tag{3.7}$$

It will be shown in the next section that the spectrum of the operator I_1 consists of the points $\lambda = q^n$, $n = 0, 1, 2, \cdots$. Thus, we see that the pair of the operators I_1 and J form a Leonard pair (see [20], where P. Terwilliger has actually introduced this notion in an effort to interpret the results of D. Leonard [21]). Recall that a pair of operators R_1 and R_2, acting in a linear space $\mathcal{L}$, is a Leonard pair if

(a) there exists a basis in $\mathcal{L}$, with respect to which the operator R_1 is diagonal, and the operator R_2 has the form of a Jacobi matrix;

(b) there exists another basis of $\mathcal{L}$, with respect to which the operator R_2 is diagonal, and the operator R_1 has the form of a Jacobi matrix.

Properties of Leonard pairs of operators in finite dimensional spaces are studied in detail (see, for example, [22] and [23]). Leonard pairs in infinite dimensional spaces are more complicated and only some isolated results are known in this case (see, for example, [23]).

4. Spectrum of I_1 and orthogonality of little q-Jacobi polynomials

The aim of this section is to find, by using the Leonard pair (I_1, J), a basis in the Hilbert space $\mathcal{H}_l$, which consists of eigenfunctions of the operator I_1 in a normalized form, and to derive explicitly the unitary matrix U, connecting this basis with the canonical basis f_n^l, $n = 0, 1, 2, \cdots$, in $\mathcal{H}_l$. This matrix directly leads the orthogonality relation for the little q-Jacobi polynomials.

A word of explanation is in order. It is a well-known fact that the little q-Jacobi polynomials (3.3) represent a particular case of the big q-Jacobi polynomials (see formula (6.3) below) with the vanishing parameter c. Therefore, it may seem that one should start directly with the latter set of polynomials – wherein the corresponding results for the former family (3.3) are then recovered by assuming that $c = 0$. However, we are interested in finding dual families with respect to polynomials (3.3) and (6.3), and establishing their properties. It turns out that an interrelation between dual little q-Jacobi and dual big q-Jacobi polynomials is more intricate than that occurring between polynomials (3.3) and (6.3) themselves (for instance,

6

the orthogonality measure for dual little q-Jacobi polynomials proves to be an extremal one, whereas the orthogonality measure for dual big q-Jacobi polynomials is not extremal). For this reason we find it more instructive to begin our discussion with the simpler case (3.3) in this section and then to consider duals of the polynomials (6.3) in sections 6–8.

Let us analyze a form of the spectrum of the operator I_1 from the point of view of the representations of the algebra $\mathrm{su}_q(1,1)$. If λ is a spectral point of the operator I_1, then (as it is easy to see from (3.7)) a successive action by the operator J upon the function (eigenfunction of I_1) ξ_λ leads to the functions

$$\xi_{q^m \lambda}, \quad m = 0, \pm 1, \pm 2, \cdots. \tag{4.1}$$

However, since I_1 is a trace class operator, not all these points can belong to the spectrum of I_1, since $q^{-m}\lambda \to \infty$ when $m \to \infty$ if $\lambda \neq 0$. This means that the coefficient $\lambda' - 1$ of $\xi_{q^{-1}\lambda'}(x)$ in (3.7) must vanish for some eigenvalue λ'. Clearly, it vanishes when $\lambda' = 1$. Moreover, this is the only possibility for vanishing a coefficient at $\xi_{q^{-1}\lambda'}(x)$ in (3.7), that is, the point $\lambda = 1$ is a spectral point for the operator I_1. Let us show that the corresponding eigenfunction $\xi_1(x) \equiv \xi_{q^0}(x)$ belongs to the representation space $\mathcal{H}_l$.

Observe that by formula (II.6) of Appendix II in [15], one has

$$p_n(1; a, b|q) = {}_2\phi_1(q^{-n}, abq^{n+1};\ aq;\ q, q) = \frac{(b^{-1}q^{-n}; q)_n}{(aq; q)_n}(abq^{n+1})^n.$$

Since $(b^{-1}q^{-n}; q)_n = (bq; q)_n(-b^{-1}q^{-1})^n q^{-n(n-1)/2}$, this means that

$$p_n(1; a, b|q) = \frac{(bq; q)_n}{(aq; q)_n}(-a)^n q^{n(n+1)/2}.$$

Therefore,

$$\langle \xi_1(x), \xi_1(x) \rangle = \sum_{n=0}^{\infty} \frac{(abq, aq; q)_n (1 - abq^{2n+1})}{(bq, q; q)_n (1 - abq)(aq)^n} p_n^2(1; a, b|q)$$

$$= \sum_{n=0}^{\infty} \frac{(abq, bq; q)_n (1 - abq^{2n+1})}{(aq, q; q)_n (1 - abq)} a^n q^{n^2} = \frac{(abq^2; q)_\infty}{(aq; q)_\infty}. \tag{4.2}$$

The last leg of this equality is obtained from formula (A.1) of Appendix. Thus, the series (4.2) converges and, therefore, the point $\lambda = 1$ actually belongs to the spectrum of the operator I_1.

Let us find other spectral points of the operator I_1 (recall that a spectrum of I_1 is discrete). Setting $\lambda = 1$ in (3.7), we see that the operator J transforms $\xi_{q^0}(x)$ into a linear combination of the functions $\xi_q(x)$ and $\xi_{q^0}(x)$. Moreover, $\xi_q(x)$ belongs to the Hilbert space $\mathcal{H}_l$, since the series

$$\langle \xi_q, \xi_q \rangle = \sum_{n=0}^{\infty} \frac{(abq, aq; q)_n (1 - abq^{2n+1})}{(bq, q; q)_n (1 - abq)(aq)^n} p_n^2(q; a, b|q) < \infty$$

is majorized by the corresponding series for $\xi_{q^0}(x)$, considered above. Therefore, $\xi_q(x)$ belongs to the Hilbert space $\mathcal{H}_l$ and the point q is an eigenvalue of the operator I_1. Similarly, setting $\lambda = q$ in (3.7), we find that $\xi_{q^2}(x)$ is an eigenfunction of I_1 and the point q^2 belongs to the spectrum of I_1. Repeating this procedure, we find that $\xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, are eigenfunctions of I_1 and the set q^n, $n = 0, 1, 2, \cdots$, belongs to the spectrum of I_1. So far, we do not know yet whether other spectral points exist or not.

The functions $\xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, are linearly independent elements of the representation space $\mathcal{H}_l$ (since they correspond to different eigenvalues of the selfadjoint operator I_1).

Suppose that values q^n, $n = 0, 1, 2, \cdots$, constitute a whole spectrum of the operator I_1. Then the set of functions $\xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, is a basis in the Hilbert space $\mathcal{H}_l$. Introducing the notation $\Xi_n := \xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, we find from (3.7) that

$$J \Xi_n = -aq^{-n}(1 - bq^{n+1}) \Xi_{n+1} + q^{-n}(a+1) \Xi_n - q^{-n}(1 - q^n) \Xi_{n-1}. \tag{4.3}$$

As we see, the matrix of the operator J in the basis Ξ_n, $n = 0, 1, 2, \cdots$, is not symmetric, although in the initial basis f_n^l, $n = 0, 1, 2, \cdots$, it was symmetric. The reason is that the matrix (a_{mn}) with entries

$$a_{mn} := \beta_m(q^n), \quad m, n = 0, 1, 2, \cdots,$$

where $\beta_m(q^n)$ are the coefficients (3.4) in the expansion $\xi_{q^n}(x) = \sum_m \beta_m(q^n) f_n^l(x)$, is not unitary. It is equivalent to the statement that the basis $\Xi_n := \xi_{q^n}(x)$, $n = 0, 1, 2, \cdots$, is not normalized. To normalize it, one has to multiply Ξ_n by corresponding numbers c_n (which are not known at this moment). Let $\hat{\Xi}_n = c_n \Xi_n$, $n = 0, 1, 2, \cdots$, be a normalized basis. Then the matrix of the operator J is symmetric in this basis. Since J has in the basis $\{\hat{\Xi}_n\}$ the form

$$J \hat{\Xi}_n = -c_{n+1}^{-1} c_n aq^{-n}(1 - bq^{n+1}) \hat{\Xi}_{n+1} + q^{-n}(a+1) \hat{\Xi}_n - c_{n-1}^{-1} c_n q^{-n}(1 - q^n) \hat{\Xi}_{n-1}, \tag{4.4}$$

then its symmetricity means that

$$c_{n+1}^{-1} c_n aq^{-n}(1 - bq^{n+1}) = c_n^{-1} c_{n+1} q^{-n-1}(1 - q^{n+1}),$$

that is, $c_n/c_{n-1} = \sqrt{aq(1 - bq^n)/(1 - q^n)}$. Therefore,

$$c_n = c(aq)^{n/2} \frac{(bq; q)_n^{1/2}}{(q; q)_n^{1/2}},$$

where c is a constant.

The expansions

$$\hat{\xi}_{q^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n \beta_m(q^n) f_m^l(x) \tag{4.5}$$

connect two orthonormal bases in the representation space $\mathcal{H}_l$. This means that the matrix $(\hat{a}_{mn})$, $m, n = 0, 1, 2, \cdots$, with entries

$$\hat{a}_{mn} = c_n \beta_m(q^n) = c \left((aq)^{n-m} \frac{(bq; q)_n}{(q; q)_n} \frac{(abq, aq; q)_m (1 - abq^{2m+1})}{(bq, q; q)_m (1 - abq)} \right)^{1/2} p_m(q^n; a, b|q) \tag{4.6}$$

is unitary, provided that the constant c is appropriately chosen. In order to calculate this constant, we use the relation $\sum_{m=0}^{\infty} |\hat{a}_{mn}|^2 = 1$ for $n = 0$. Then this sum is a multiple of the sum in (4.2) and, consequently,

$$c = \frac{(aq; q)_\infty^{1/2}}{(abq; q)_\infty^{1/2}}.$$

Thus the c_n in (4.5) and (4.6) is real and equals to

$$c_n = \left(\frac{(aq; q)_\infty}{(abq; q)_\infty} \frac{(bq; q)_n (aq)^n}{(q; q)_n} \right)^{1/2}.$$

8

The matrix $(\hat{a}_{mn})$ is orthogonal, that is,

$$\sum_n \hat{a}_{mn}\hat{a}_{m'n} = \delta_{mm'}, \qquad \sum_m \hat{a}_{mn}\hat{a}_{mn'} = \delta_{nn'}. \tag{4.7}$$

Substituting into the first sum over n in (4.7) the expressions for $\hat{a}_{mn}$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n(aq)^n}{(q;q)_n} \, p_m(q^n;a,b|q) \, p_{m'}(q^n;a,b|q)$$

$$= \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(1-abq)(aq)^m \, (bq,q;q)_m}{(1-abq^{2m+1})\,(abq,aq;q)_m} \, \delta_{mm'}, \tag{4.8}$$

which must yield the orthogonality relation for the little q-Jacobi polynomials. An only gap, which appears here, is the following. We have assumed that the points q^n, $n = 0, 1, 2, \cdots$, exhaust the whole spectrum of the operator I_1. Let us show that this is the case.

Recall that the selfadjoint operator I_1 is represented by a Jacobi matrix in the basis $f_n^l(x)$, $n = 0, 1, 2, \cdots$. According to the theory of operators of such type (see, for example, [18], Chapter VII), eigenfunctions ξ_λ of I_1 are expanded into series in the monomials $f_n^l(x)$, $n = 0, 1, 2, \cdots$, with coefficients, which are polynomials in λ. These polynomials are orthogonal with respect to some positive measure $d\mu(\lambda)$ (moreover, for selfadjoint operators this measure is unique). The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator under consideration and the spectrum is simple. Let us apply these assertions to the operator I_1.

We have found that the spectrum of I_1 contains the points q^n, $n = 0, 1, 2, \cdots$. If the operator I_1 would have other spectral points x, then on the left-hand side of (4.8) there would be other summands $\mu_{x_k} p_m(x_k;a,b|q) p_{m'}(x_k;a,b|q)$, corresponding to these additional points. Let us show that these additional summands do not appear. To this end we set $m = m' = 0$ in the relation (4.8) with the additional summands. Since $p_0(x;a,b|q) = 1$, we have the equality

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n(aq)^n}{(q;q)_n} + \sum_k \mu_{x_k} = \frac{(abq^2;q)_\infty}{(aq;q)_\infty}.$$

According to the q-binomial theorem (see formula (1.3.2) in [15]), we have

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n \, (aq)^n}{(q;q)_n} = \frac{(abq^2;q)_\infty}{(aq;q)_\infty}. \tag{4.9}$$

Hence, $\sum_k \mu_{x_k} = 0$ and all μ_{x_k} disappear. This means that additional summands do not appear in (4.8) and it does represent the orthogonality relation for the little q-Jacobi polynomials.

By using the operators I_1 and J, which form a Leonard pair, we thus derived the orthogonality relation for little q-Jacobi polynomials.

The orthogonality relation for the little q-Jacobi polynomials is given by formula (4.8). Due to this orthogonality, we arrive at the following statement: *The spectrum of the operator I_1 coincides with the set of points q^n, $n = 0, 1, 2, \cdots$. The spectrum is simple and has one accumulation point at 0.*

5. Dual little q-Jacobi polynomials

Now we consider the second identity in (4.7), which gives the orthogonality relation for the matrix elements $\hat{a}_{mn}$, considered as functions of m. Up to multiplicative factors these functions coincide with the functions

$$F_n(x; a, b|q) = {}_2\phi_1(x, abq/x; \; aq; \; q, q^{n+1}), \tag{5.1}$$

considered on the set $x \in \{q^{-m} \,|\, m = 0, 1, 2, \cdots\}$. Consequently,

$$\hat{a}_{mn} = \left(\frac{(aq; q)_\infty}{(abq; q)_\infty} \frac{(bq; q)_n}{(q; q)_n} (aq)^{n-m} \frac{(abq, aq; q)_m (1 - abq^{2m+1})}{(bq, q; q)_m} \right)^{1/2} F_n(q^{-m}; a, b|q)$$

and the second identity in (4.7) gives the orthogonality relation for the functions (5.1):

$$\sum_{m=0}^{\infty} \frac{(1 - abq^{2m+1})(abq, aq; q)_m}{(1 - abq)(aq)^m (bq, q; q)_m} F_n(q^{-m}; a, b|q) \, F_{n'}(q^{-m}; a, b|q)$$

$$= \frac{(abq^2; q)_\infty}{(aq; q)_\infty} \frac{(q; q)_n (aq)^{-n}}{(bq; q)_n} \delta_{nn'}. \tag{5.2}$$

The functions $F_n(x; a, b|q)$ can be represented in another form. Indeed, one can use the relation (III.8) of Appendix III in [15] in order to obtain that

$$F_n(q^{-m}; a, b|q) = \frac{(b^{-1}q^{-m}; q)_m}{(aq; q)_m} (abq^{m+1})^m \, {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n}; \; bq; \; q, q^n/a)$$

$$= \frac{(-1)^m (bq; q)_m}{(aq; q)_m} a^m \, q^{m(m+1)/2} \, {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n}; \; bq; \; q, q^n/a). \tag{5.3}$$

The basic hypergeometric function ${}_3\phi_1$ in (5.3) is a polynomial of degree n in the variable $\mu(m) := q^{-m} + ab\, q^{m+1}$, which represents a q-quadratic lattice; we denote it

$$d_n(\mu(m); a, b|q) := {}_3\phi_1(q^{-m}, ab\, q^{m+1}, q^{-n}; \; bq; \; q, q^n/a). \tag{5.4}$$

Then formula (5.2) yields the orthogonality relation

$$\sum_{m=0}^{\infty} \frac{(1 - abq^{2m+1})(abq, bq; q)_m}{(1 - abq)(aq, q; q)_m} a^m \, q^{m^2} \, d_n(\mu(m)) \, d_{n'}(\mu(m))$$

$$= \frac{(abq^2; q)_\infty}{(aq; q)_\infty} \frac{(q; q)_n (aq)^{-n}}{(bq; q)_n} \delta_{nn'} \tag{5.5}$$

for the polynomials (5.4). We call the polynomials $d_n(\mu(m); a, b|q)$ *dual little q-Jacobi polynomials*. Note that these polynomials can be expressed in terms of the Al-Salam–Chihara polynomials

$$Q_n(x; a, b|q) = \frac{(ab; q)_n}{a^n} \, {}_3\phi_2\left(\begin{array}{c} q^{-n}, az, az^{-1} \\ ab, 0 \end{array} \bigg| \, q, q \right), \quad x = \frac{1}{2}(z + z^{-1}),$$

with the parameter $q > 1$. An explicit relation between them is

$$d_n(\mu(x); \beta/\alpha, \, 1/\alpha\beta q \,|\, q) = \frac{q^{n(n-1)/2}}{(-\beta)^n (1/\alpha\beta; q)_n} Q_n(\alpha\mu(x)/2; \alpha, \beta|q^{-1}).$$

10

Ch. Berg and M. E. H. Ismail studied this type of Al-Salam–Chihara polynomials in [24] and derived continuous complex orthogonality measures for them. But [24] does not contain any discussion of the duality of this family of polynomials with respect to little q-Jacobi polynomials.

A recurrence relation for the polynomials $d_n(\mu(m); a, b|q)$ is derived from formula (3.6). It has the form

$$(q^{-m} + abq^{m+1}) d_n(\mu(m)) = -a q^{-n}(1 - bq^{n+1}) d_{n+1}(\mu(m))$$

$$+ q^{-n}(1 + a) d_n(\mu(m)) - q^{-n}(1 - q^n) d_{n-1}(\mu(m)),$$

where $d_n(\mu(m)) \equiv d_n(\mu(m); a, b|q)$. Comparing this relation with the recurrence relation (3.69) in [25], we see that the polynomials (5.4) are multiple to the polynomials (3.67) in [25]. Moreover, if one takes into account this multiplicative factor, the orthogonality relation (5.5) for polynomials (5.4) turns into relation (3.82) for the polynomials (3.67) in [25], although the derivation of the orthogonality relation in [25] is more complicated than our derivation of (5.5). The authors of [25] do not give an explicit form of their polynomials in the form similar to (5.4). Concerning the polynomials (3.67) in [25] see also [26].

Let $\mathfrak{l}^2$ be the Hilbert space of functions on the set $m = 0, 1, 2, \cdots$ with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \frac{(1 - abq^{2m+1}) (abq, bq; q)_m}{(1 - abq) (aq, q; q)_m} a^m q^{m^2} f_1(m) \overline{f_2(m)}.$$

The polynomials (5.4) are in one-to-one correspondence with the columns of the unitary matrix $(\hat{a}_{mn})$ and the orthogonality relation (5.5) is equivalent to the orthogonality of these columns. Due to (4.7) the columns of the matrix $(\hat{a}_{mn})$ form an orthonormal basis in the Hilbert space of sequences $\mathbf{a} = \{a_n \,|\, n = 0, 1, 2, \cdots\}$ with the scalar product $\langle \mathbf{a}, \mathbf{a}' \rangle = \sum_n a_n a'_n$. For this reason, the set of polynomials $d_n(\mu(m); a, b|q)$, $n = 0, 1, 2, \cdots$, form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$. This means that *the point measure in (5.5) is extremal for the dual little q-Jacobi polynomials $d_n(\mu(m); a, b|q)$* (for the definition of an extremal orthogonality measure see, for example, [27]).

6. Pair of operators (I_2, J)

Let b and c be real numbers such that $0 < b < q^{-1}$ and $c < 0$. We consider the operator

$$I_2 := \alpha q^{J_0/4}(J_+ A + A J_-) q^{J_0/4} - B \tag{6.1}$$

of the representation T_l^+, where $\alpha = (-ac/q)^{1/2} (1 - q)$, $a = q^{2l-1}$, and A and B are operators of the form

$$A = \frac{q^{(J_0-l+2)/2}\sqrt{(1 - bq^{J_0-l+1})(1 - abq^{J_0-l+1})(1 - cq^{J_0-l+1})(1 - abc^{-1}q^{J_0-l+1})}}{(1 - abq^{2J_0-2l+2})\sqrt{(1 - abq^{2J_0-2l+1})(1 - abq^{2J_0-2l+3})}},$$

$$B = \frac{(1 - aq^{J_0-l+1})(1 - abq^{J_0-l+1})(1 - cq^{J_0-l+1})}{(1 - abq^{2J_0-2l+1})(1 - abq^{2J_0-2l+2})}$$

$$-ac\, q^{J_0-l+1} \frac{(1 - q^{J_0-l})(1 - bq^{J_0-l})(1 - abc^{-1}q^{J_0-l})}{(1 - abq^{2J_0-2l})(1 - abq^{2J_0-2l+1})} - 1.$$

We have the following formula for the symmetric operator I_2:

$$I_2 f_n^l = a_n f_{n+1}^l + a_{n-1} f_{n-1}^l - b_n f_n^l, \tag{6.2}$$

11

where

$$a_{n-1} = (-acq^{n+1})^{1/2} \frac{\sqrt{(1-q^n)(1-aq^n)(1-bq^n)(1-abq^n)(1-cq^n)(1-abc^{-1}q^n)}}{(1-abq^{2n})\sqrt{(1-abq^{2n-1})(1-abq^{2n+1})}},$$

$$b_n = \frac{(1-aq^{n+1})(1-abq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})} - acq^{n+1}\frac{(1-q^n)(1-bq^n)(1-abq^n/c)}{(1-abq^{2n})(1-abq^{2n+1})} - 1,$$

with $a = q^{2l-1}$. Recall that l takes any positive value, so a can be any positive number such that $a < q^{-1}$. Since the $q^{\pm J_0}$ are symmetric operators, the operator I_2 is also symmetric.

The operator I_2 is bounded. Therefore, we assume that it is defined on the whole representation space $\mathcal{H}_l$. This means that I_2 is a selfadjoint operator. Actually, I_2 is a trace class operator. To show this we note that for the coefficients a_n and b_n from (6.2) one obtains that

$$a_{n+1}/a_n \to q^{1/2}, \quad b_{n+1}/b_n \to q \quad \text{when} \quad n \to \infty.$$

Therefore, $\sum_n (2a_n + b_n) < \infty$ and this means that I_2 is a trace class operator. Thus, the spectrum of I_2 is simple (since it is representable by a Jacobi matrix with $a_n \neq 0$), discrete and have a single accumulation point at 0.

To find eigenfunctions $\psi_\lambda(x)$ of the operator I_2, $I_2\psi_\lambda(x) = \lambda\psi_\lambda(x)$, we set

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \beta_n(\lambda)f_n^l(x).$$

Acting by the operator I_2 on both sides of this relation, one derives that

$$\sum_n \beta_n(\lambda)(a_n f_{n+1}^l + a_{n-1}f_{n-1}^l - b_n f_n^l) = \lambda \sum \beta_n(\lambda)f_n^l,$$

where a_n and b_n are the same as in (6.2). Collecting in this identity factors, which multiply f_n^l with fixed n, we arrive at the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$a_n\beta_{n+1}(\lambda) + a_{n-1}\beta_{n-1}(\lambda) - b_n\beta_n(\lambda) = \lambda\beta_n(\lambda).$$

Making the substitution

$$\beta_n(\lambda) = \left(\frac{(abq, aq, cq; q)_n (1-abq^{2n+1})}{(abq/c, bq, q; q)_n (1-abq)(-ac)^n}\right)^{1/2} q^{-n(n+3)/4} \beta_n'(\lambda)$$

we reduce this relation to the following one

$$A_n\beta_{n+1}'(\lambda) + C_n\beta_{n-1}'(\lambda) - (A_n + C_n - 1)p_n'(\lambda) = \lambda\beta_n'(\lambda)$$

with

$$A_n = \frac{(1-aq^{n+1})(1-cq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$

$$C_n = \frac{-acq^{n+1}(1-q^n)(1-bq^n)(1-abc^{-1}q^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

It is the recurrence relation for the big q-Jacobi polynomials

$$P_n(\lambda; a, b, c; q) := {}_3\phi_2(q^{-n}, abq^{n+1}, \lambda; aq, cq; q, q) \qquad (6.3)$$

introduced by G. E. Andrews and R. Askey [28] (see also formula (7.3.10) in [15]). Therefore, $\beta'_n(\lambda) = P_n(\lambda; a, b, c; q)$ and

$$\beta_n(\lambda) = \left(\frac{(abq, aq, cq; q)_n (1 - abq^{2n+1})}{(abq/c, bq, q; q)_n (1 - abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4} P_n(\lambda; a, b, c; q). \qquad (6.4)$$

For the eigenfunctions $\psi_\lambda(x)$ we have the expansion

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \left(\frac{(abq, aq, cq; q)_n (1 - abq^{2n+1})}{(abq/c, bq, q; q)_n (1 - abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4} P_n(\lambda; a, b, c; q) f_n^l(x)$$

$$= \sum_{n=0}^{\infty} a^{-n/4} \frac{(aq; q)_n}{(q; q)_n} \left(\frac{(abq, cq; q)_n (1 - abq^{2n+1})}{(abq/c, bq; q)_n (1 - abq)(-ac)^n} \right)^{1/2} q^{-n(n+3)/4} P_n(\lambda; a, b, c; q) x^n.$$
$$(6.5)$$

Since the spectrum of the operator I_2 is discrete, only a discrete set of these functions belongs to the Hilbert space $\mathcal{H}_l$ and this discrete set determines the spectrum of I_2.

In what follows we intend to study a spectrum of the operator I_2 and to find polynomials, dual to big q-Jacobi polynomials. It can be done with the aid of the operator

$$J := q^{-J_0+l} + ab\, q^{J_0-l+1} \equiv \mu(J_0 - l),$$

which has been already used in the previous case in section 3. In order to determine how this operator acts upon the eigenfunctions $\psi_\lambda(x)$, one can use the q-difference equation

$$(q^{-n} + abq^{n+1}) P_n(\lambda) = aq\lambda^{-2}(\lambda - 1)(b\lambda - c) P_n(q\lambda)$$

$$-[\lambda^{-2}acq(1 + q) - \lambda^{-1}q(ab + ac + a + c)] P_n(\lambda) + \lambda^{-2}(\lambda - aq)(\lambda - cq) P_n(q^{-1}\lambda), \qquad (6.6)$$

for the big q-Jacobi polynomials $P_n(\lambda) \equiv P_n(\lambda; a, b, c; q)$ (see, for example, formula (3.5.5) in [19]). Multiply both sides of (6.6) by $d_n f_n^l(x)$, where d_n are the coefficients of $P_n(\lambda; a, b, c; q)$ in the expression (6.4) for the coefficients $\beta_n(\lambda)$, and sum over n. Taking into account formula (6.5) and the fact that $J f_n^l(x) = (q^{-n} + ab\, q^{n+1}) f_n^l(x)$, one obtains the relation

$$J \psi_\lambda(x) = aq\lambda^{-2}(\lambda - 1)(b\lambda - c) \psi_{q\lambda}(x)$$

$$-[\lambda^{-2}acq(1 + q) - \lambda^{-1}q(ab + ac + a + c)] \psi_\lambda(x) + \lambda^{-2}(\lambda - aq)(\lambda - cq) \psi_{q^{-1}\lambda}(x). \qquad (6.7)$$

It will be shown in the next section that the spectrum of the operator I_2 consists of the points aq^n, cq^n, $n = 0, 1, 2, \cdots$. The matrix J consists of two Jacobi matrices (one corresponds to the basis elements aq^n, $n = 0, 1, 2, \cdots$, and another to the basis elements cq^n, $n = 0, 1, 2, \cdots$). In this case, the operators I_2 and J form some generalization of a Leonard pair.

7. Spectrum of I_2 and orthogonality of big q-Jacobi polynomials

As in section 4 one can show that for some value of λ (which must belong to the spectrum) the last term on the right side of (6.7) has to vanish. There are two such values of λ: $\lambda = aq$ and $\lambda = cq$. Let us show that both of these points are spectral points of the operator I_2. Observe that, according to (6.3),

$$P_n(aq; a, b, c; q) := {}_2\phi_1(q^{-n}, abq^{n+1}; cq; q, q) = \frac{(c/abq^n; q)_n}{(cq; q)_n}(ab)^n q^{n(n+1)}.$$

13

Therefore, since
$$(c/abq^n;q)_n = (abq/c;q)_n(-c/ab)^n q^{-n(n+1)/2},$$

one obtains that
$$P_n(aq;a,b,c;q) := \frac{(abq/c;q)_n}{(cq;q)_n}(-c)^n q^{n(n+1)/2}. \tag{7.1}$$

Likewise,
$$P_n(cq;a,b,c;q) := \frac{(bq;q)_n}{(aq;q)_n}(-a)^n q^{n(n+1)/2}.$$

Hence, for the scalar product $\langle \psi_{aq}(x), \psi_{aq}(x) \rangle$ we have the expression

$$\sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq,aq,cq;q)_n}{(1-abq)(abq/c,bq,q;q)_n(-ac)^n} q^{-n(n+3)/2} P_n^2(aq;a,b,c;q)$$

$$= \sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq/c,abq,aq;q)_n}{(1-abq)(bq,cq,q;q)_n(-a/c)^n} q^{n(n-1)/2} = \frac{(abq^2,c/a;q)_\infty}{(bq,cq;q)_\infty}, \tag{7.2}$$

where the relation (A.6) from Appendix has been used. Similarly, for $\langle \psi_{cq}(x), \psi_{cq}(x) \rangle$ one has the expression

$$\sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq,aq,cq;q)_n}{(1-abq)(-ac)^n(abq/c,bq,q;q)_n} q^{-n(n+3)/2} P_n^2(cq;a,b,c;q) = \frac{(abq^2,a/c;q)_\infty}{(aq,abq/c;q)_\infty}, \tag{7.3}$$

where formula (A.7) from Appendix has been used. Thus, the values $\lambda = aq$ and $\lambda = cq$ are the spectral points of the operator I_2.

Let us find other spectral points of the operator I_2. Setting $\lambda = aq$ in (6.7), we see that the operator J transforms $\psi_{aq}(x)$ into a linear combination of the functions $\psi_{aq^2}(x)$ and $\psi_{aq}(x)$. We have to show that $\psi_{aq^2}(x)$ belongs to the Hilbert space $\mathcal{H}_l$, that is, that

$$\langle \psi_{aq^2}, \psi_{aq^2} \rangle = \sum_{n=0}^{\infty} \frac{(abq,aq,cq;q)_n(1-abq^{2n+1})}{(abq/c,bq,q;q)_n(1-abq)(-ac)^n} q^{-n(n+3)/2} P_n^2(aq^2;a,b,c;q) < \infty.$$

It is harder to prove this inequality directly, than in the case of the little q-Jacobi polynomials. Therefore we do not want to embark on the discussion of this point here (for this inequality actually follows from the orthogonality relation for the big q-Jacobi polynomials). The above inequality shows that $\psi_{aq^2}(x)$ is an eigenfunction of I_2 and the point aq^2 belongs to the spectrum of the operator I_2. Setting $\lambda = aq^2$ in (6.7) and acting similarly, one obtains that $\psi_{aq^3}(x)$ is an eigenfunction of I_2 and the point aq^3 belongs to the spectrum of I_2. Repeating this procedure, one sees that $\psi_{aq^n}(x)$, $n = 1, 2, \cdots$, are eigenfunctions of I_2 and the set aq^n, $n = 1, 2, \cdots$, belongs to the spectrum of I_2. Likewise, one concludes that $\psi_{cq^n}(x)$, $n = 1, 2, \cdots$, are eigenfunctions of I_2 and the set cq^n, $n = 1, 2, \cdots$, belongs to the spectrum of I_2. Note that so far we do not know whether the operator I_2 has other spectral points or not. In order to solve this problem we shall proceed as in section 4.

The functions $\psi_{aq^n}(x)$ and $\psi_{cq^n}(x)$, $n = 1, 2, \cdots$, are linearly independent elements of the representation space $\mathcal{H}_l$. Suppose that aq^n and cq^n, $n = 1, 2, \cdots$, constitute the whole spectrum of the operator I_2. Then the set of functions $\psi_{aq^n}(x)$ and $\psi_{cq^n}(x)$, $n = 1, 2, \cdots$, is a basis of the Hilbert space $\mathcal{H}_l$. Introducing the notations $\Xi_n := \xi_{aq^{n+1}}(x)$ and $\Xi'_n := \xi_{cq^{n+1}}(x)$, $n = 0, 1, 2, \cdots$, we find from (6.7) that

$$J\Xi_n = a^{-1}cq^{-2n-1}(1-aq^{n+1})(1-baq^{n+1}/c)\Xi_{n+1} + d_n\Xi_n + a^{-1}cq^{-2n}(1-q^n)(1-aq^n/c)\Xi_{n-1},$$

$$J\,\Xi'_n = c^{-1}aq^{-2n-1}(1-cq^{n+1})(1-bq^{n+1})\,\Xi_{n+1} + d'_n\,\Xi_n + c^{-1}aq^{-2n}(1-q^n)(1-cq^n/a)\,\Xi_{n-1},$$

where

$$d_n = \frac{1}{a}[q^{-2n-1}c(1+q) - q^{-n}(ab+ac+a+c)],$$

$$d'_n = \frac{1}{c}[q^{-2n-1}a(1+q) - q^{-n}(ab+ac+a+c)].$$

As we see, the matrix of the operator J in the basis $\Xi_n = \xi_{aq^{n+1}}(x)$, $\Xi'_n = \xi_{cq^{n+1}}(x)$, $n = 0,1,2,\cdots$, is not symmetric, although in the initial basis f^l_n, $n = 0,1,2,\cdots$, it was symmetric. The reason is that the matrix $M := (a_{mn} \ a'_{mn})$ with entries

$$a_{mn} := \beta_m(aq^n), \quad a'_{mn} := \beta_m(cq^n), \quad m,n = 0,1,2,\cdots,$$

where $\beta_m(dq^n)$, $d = a,c$, are coefficients (6.4) in the expansion $\psi_{dq^n}(x) = \sum_m \beta_m(dq^n) f^l_n(x)$ (see above), is not unitary. This matrix M is formed by adding the columns of the matrix (a'_{mn}) to the columns of the matrix (a_{mn}) from the right, that is,

$$M = \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a'_{11} & \cdots & a'_{1l} & \cdots \\ a_{21} & \cdots & a_{2k} & \cdots & a'_{21} & \cdots & a'_{2l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j1} & \cdots & a_{jk} & \cdots & a'_{j1} & \cdots & a'_{jl} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It maps the basis $\{f^l_n\}$ into the basis $\{\psi_{aq^{n+1}}, \psi_{cq^{n+1}}\}$ in the representation space. The nonunitarity of the matrix M is equivalent to the statement that the basis $\Xi_n := \xi_{aq^{n+1}}(x)$, $\Xi_n := \xi_{cq^{n+1}}(x)$, $n = 0,1,2,\cdots$, is not normalized. In order to normalize it we have to multiply Ξ_n by appropriate numbers c_n and Ξ'_n by numbers c'_n. Let $\hat{\Xi}_n = c_n\Xi_n$, $\hat{\Xi}'_n = c'_n\Xi_n$, $n = 0,1,2,\cdots$, be a normalized basis. Then the operator J is symmetric in this basis and has the form

$$J\,\hat{\Xi}_n = c^{-1}_{n+1}c_n a^{-1}cq^{-2n-1}(1-aq^{n+1})(1-abq^{n+1}/c)\,\hat{\Xi}_{n+1} + d_n\,\hat{\Xi}_n$$

$$+c^{-1}_{n-1}c_n a^{-1}cq^{-2n}(1-aq^n/c)(1-q^n)\,\hat{\Xi}_{n-1}, \tag{7.4}$$

$$J\,\hat{\Xi}'_n = c'^{-1}_{n+1}c'_n c^{-1}aq^{-2n-1}(1-bq^{n+1})(1-cq^{n+1})\,\hat{\Xi}_{n+1} + d'_n\,\hat{\Xi}_n$$

$$-c'^{-1}_{n-1}c'_n c^{-1}aq^{-2n}(1-cq^n/a)(1-q^n)\,\hat{\Xi}_{n-1}, \tag{7.5}$$

The symmetricity of the matrix of the operator J in the basis $\{\hat{\Xi}_n, \hat{\Xi}'_n\}$ means that

$$c^{-1}_{n+1}c_n q^{-2n-1}(1-aq^{n+1})(1-abq^{n+1}/c) = c^{-1}_n c_{n+1}q^{-2n-2}(1-aq^{n+1}/c)(1-q^{n+1}),$$

$$c'^{-1}_{n+1}c'_n q^{-2n-1}(1-bq^{n+1})(1-cq^{n+1}) = c'^{-1}_n c'_{n+1}q^{-2n-2}(1-cq^{n+1}/a)(1-q^{n+1}).$$

that is,

$$\frac{c_n}{c_{n-1}} = \sqrt{q\frac{(1-aq^n)(1-abq^n/c)}{(1-q^n)(1-aq^n/c)}}, \quad \frac{c'_n}{c'_{n-1}} = \sqrt{q\frac{(1-cq^n)(1-bq^n)}{(1-q^n)(1-cq^n/a)}}.$$

Thus,

$$c_n = C\left(q^n\frac{(abq/c, aq; q)_n}{(aq/c, q; q)_n}\right)^{1/2}, \quad c'_n = C'\left(q^n\frac{(bq, cq; q)_n}{(cq/a, q; q)_n}\right)^{1/2},$$

where C and C' are some constants.

Therefore, in the expansions

$$\hat{\psi}_{aq^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n\, \beta_m(aq^n)\, f_m^l(x) = \sum_m \hat{a}_{mn}\, f_m^l(x), \tag{7.6}$$

$$\hat{\psi}_{cq^n}(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n'\, \beta_m(cq^n)\, f_m^l(x) = \sum_m \hat{a}_{mn}'\, f_m^l(x), \tag{7.7}$$

the matrix $\hat{M} := (\hat{a}_{mn}\ \hat{a}_{mn}')$ with entries

$$\hat{a}_{mn} = c_n\, \beta_m(aq^n) = C \left(q^n \frac{(abq/c, aq; q)_n}{(aq/c, q; q)_n} \frac{(abq, aq, cq; q)_m\, (1 - abq^{2m+1})}{(abq/c, bq, q; q)_m\, (1 - abq)(-ac)^m} \right)^{1/2}$$

$$\times q^{-m(m+3)/4}\, P_m(aq^{n+1}; a, b, c; q), \tag{7.8}$$

$$\hat{a}_{mn}' = c_n\, \beta_m(cq^n) = C' \left(q^n \frac{(bq, cq; q)_n}{((cq/a, q; q)_n} \frac{(abq, aq, cq; q)_m\, (1 - abq^{2m+1})}{(abq/c, bq, q; q)_m\, (1 - abq)(-ac)^m} \right)^{1/2}$$

$$\times q^{-m(m+3)/4}\, P_m(cq^{n+1}; a, b, c; q), \tag{7.9}$$

is unitary, provided that the constants C and C' are appropriately chosen. In order to calculate these constants, one can use the relations

$$\sum_{m=0}^{\infty} |\hat{a}_{mn}|^2 = 1, \quad \sum_{m=0}^{\infty} |\hat{a}_{mn}'|^2 = 1,$$

for $n = 0$. Then these sums are multiples of the sums in (7.2) and (7.3), so we find that

$$C = \frac{(bq, cq; q)_\infty^{1/2}}{(abq^2, c/a; q)_\infty^{1/2}}, \quad C' = \frac{(aq, abq/c; q)_\infty^{1/2}}{(abq^2, a/c; q)_\infty^{1/2}}. \tag{7.10}$$

The coefficients c_n and c_n' in (7.6)–(7.9) are thus real and equal to

$$c_n = \left(\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \frac{(abq/c, aq; q)_n\, q^n}{(aq/c, q; q)_n} \right)^{1/2}, \quad c_n' = \left(\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \frac{(bq, cq; q)_n\, q^n}{(cq/a, q; q)_n} \right)^{1/2}.$$

The orthogonality of the matrix $\hat{M} \equiv (\hat{a}_{mn}\ \hat{a}_{mn}')$ means that

$$\sum_m \hat{a}_{mn}\hat{a}_{mn'} = \delta_{nn'}, \quad \sum_m \hat{a}_{mn}'\hat{a}_{mn'}' = \delta_{nn'}, \quad \sum_m \hat{a}_{mn}\hat{a}_{mn'}' = 0, \tag{7.11}$$

$$\sum_n (\hat{a}_{mn}\hat{a}_{m'n} + \hat{a}_{mn}'\hat{a}_{m'n}') = \delta_{mm'}. \tag{7.12}$$

Substituting the expressions for $\hat{a}_{mn}$ and $\hat{a}_{mn}'$ into (7.12), one obtains the relation

$$\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \sum_{n=0}^{\infty} \frac{(aq, abq/c; q)_n q^n}{(aq/c, q; q)_n} P_m(aq^{n+1}) P_{m'}(aq^{n+1})$$

$$+ \frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \sum_{n=0}^{\infty} \frac{(bq, cq; q)_n q^n}{(cq/a, q; q)_n} P_m(cq^{n+1}) P_{m'}(cq^{n+1})$$

16

$$= \frac{(1 - abq)(bq, abq/c, q; q)_m}{(1 - abq^{2m+1})(aq, abq, cq; q)_m} (-ac)^m q^{m(m+3)/2} \, \delta_{mm'} \,. \tag{7.13}$$

This identity must give an orthogonality relation for the big q-Jacobi polynomials $P_m(y) \equiv P_m(y; a, b, c; q)$. An only gap, which appears here, is the following. We have assumed that the points aq^n and cq^n, $n = 0, 1, 2, \cdots$, exhaust the whole spectrum of the operator I_2. As in the case of the operator I_1 in section 4, if the operator I_2 would have other spectral points x_k, then on the left-hand side of (7.13) would appear other summands $\mu_{x_k} P_m(x_k; a, b, c; q) P_{m'}(x_k; a, b, c; q)$, which correspond to these additional points. Let us show that these additional summands do not appear. To this end we set $m = m' = 0$ in the relation (7.13) with the additional summands. This results in the equality

$$\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \sum_{n=0}^{\infty} \frac{(aq, abq/c; q)_n q^n}{(aq/c, q; q)_n}$$

$$+ \frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \sum_{n=0}^{\infty} \frac{(bq, cq; q)_n q^n}{(cq/a, q; q)_n} + \sum_k \mu_{x_k} = 1. \tag{7.14}$$

In order to show that $\sum_k \mu_{x_k} = 0$, take into account the relation

$$\frac{(Aq/C, Bq/C; q)_\infty}{(q/C, ABq/C; q)_\infty} \, {}_2\phi_1(A, B; C; q, q)$$

$$\frac{(A, B; q)_\infty}{(C/q, ABq/C; q)_\infty} \, {}_2\phi_1(Aq/C, Bq/C; q^2/C; q, q) = 1$$

(see formula (2.10.13) in [15]). Putting here $A = aq$, $B = abq/c$ and $C = aq/c$, we obtain relation (7.14) without the summand $\sum_k \mu_{x_k}$. Therefore, in (7.14) the sum $\sum_k \mu_{x_k}$ does really vanish and formula (7.13) gives an orthogonality relation for big q-Jacobi polynomials.

By using the operators I_2 and J, we thus derived the orthogonality relation for big q-Jacobi polynomials.

The orthogonality relation (7.13) for big q-Jacobi polynomials enables us to formulate the following statement: *The spectrum of the operator I_2 coincides with the set of points aq^{n+1} and cq^{n+1}, $n = 0, 1, 2, \cdots$. The spectrum is simple and has one accumulation point at 0.*

8. Dual big q-Jacobi polynomials

Now we consider the relations (7.11). They give the orthogonality relation for the set of matrix elements $\hat{a}_{mn}$ and $\hat{a}'_{mn}$, viewed as functions of m. Up to multiplicative factors, they coincide with the functions

$$F_n(x; a, b, c; q) := {}_3\phi_2(x, abq/x, aq^{n+1}; aq, cq; q, q), \quad n = 0, 1, 2, \cdots, \tag{8.1}$$

$$F'_n(x; a, b, c; q) := {}_3\phi_2(x, abq/x, cq^{n+1}; aq, cq; q, q) \equiv F_n(x; c, ab/c, a), \quad n = 0, 1, 2, \cdots, \tag{8.2}$$

considered on the corresponding sets of points. Namely, we have

$$\hat{a}_{mn} \equiv \hat{a}_{mn}(a, b, c) = C \left(q^n \frac{(abq/c, aq; q)_n}{(aq/c, q; q)_n} \frac{(abq, aq, cq; q)_m (1 - abq^{2m+1})}{(abq/c, bq, q; q)_m (1 - abq)(-ac)^m} \right)^{1/2}$$

$$\times q^{-m(m+3)/4} F_n(q^{-m}; a, b, c; q), \tag{8.3}$$

$$\hat{a}'_{mn} \equiv \hat{a}'_{mn}(a,b,c) = C' \left(q^n \frac{(bq,cq;q)_n}{(cq/a,q;q)_n} \frac{(abq,aq,cq;q)_m (1-abq^{2m+1})}{(abq/c,bq,q;q)_m (1-abq)(-ac)^m} \right)^{1/2}$$

$$\times q^{-m(m+3)/4} F'_n(q^{-m};a,b,c;q) \equiv \hat{a}_{mn}(c,ab/c,a), \tag{8.4}$$

where C and C' are given by formulas (7.10). The relations (7.11) lead to the following orthogonality relation for the functions (8.1) and (8.2):

$$\frac{(bq,cq;q)_\infty}{(abq^2,c/a;q)_\infty} \sum_{m=0}^\infty \rho(m) F_n(q^{-m};a,b,c;q) F_{n'}(q^{-m};a,b,c;q) = \frac{(aq/c,q;q)_n}{(aq,abq/c;q)_n q^n} \delta_{nn'}, \tag{8.5}$$

$$\frac{(aq,abq/c;q)_\infty}{(abq^2,a/c;q)_\infty} \sum_{m=0}^\infty \rho(m) F'_n(q^{-m};a,b,c;q) F'_{n'}(q^{-m};a,b,c;q) = \frac{(cq/a,q;q)_n}{(bq,cq;q)_n q^n} \delta_{nn'}, \tag{8.6}$$

$$\sum_{m=0}^\infty \rho(m) F_n(q^{-m};a,b,c;q) F'_{n'}(q^{-m};a,b,c;q) = 0, \tag{8.7}$$

where

$$\rho(m) := \frac{(1-abq^{2m+1})(aq,abq,cq;q)_m}{(1-abq)(bq,abq/c,q;q)_m (-ac)^m} q^{-m(m+3)/2}.$$

There is another form for the functions $F_n(q^{-m};a,b,c;q)$ and $F'_n(q^{-m};a,b,c;q)$. Indeed, one can use the relation (III.12) of Appendix III in [15] to obtain that

$$F_n(q^{-m};a,b,c;q) = \frac{(cq^{-m}/ab;q)_m}{(cq;q)_m}(abq^{m+1})^m \, {}_3\phi_2 \left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| q, aq^{n+1}/c \right)$$

$$= \frac{(abq/c;q)_m}{(cq;q)_m}(-c)^m q^{m(m+1)/2} \, {}_3\phi_2 \left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| q, aq^{n+1}/c \right)$$

and

$$F'_n(q^{-m};a,b,c;q) = \frac{(bq;q)_m}{(aq;q)_m}(-a)^m q^{m(m+1)/2} \, {}_3\phi_2 \left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ bq, cq \end{matrix} \middle| q, cq^{n+1}/a \right).$$

The basic hypergeometric functions ${}_3\phi_2$ in these formulas are polynomials in $\mu(m) := q^{-m} + ab\,q^{m+1}$. So if we introduce the notation

$$D_n(\mu(m);a,b,c|q) := {}_3\phi_2 \left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| q, aq^{n+1}/c \right), \tag{8.8}$$

then

$$F_n(q^{-m};a,b,c;q) = \frac{(abq/c;q)_m}{(cq;q)_m}(-c)^m q^{m(m+1)/2} D_n(\mu(m);a,b,c|q),$$

$$F'_n(q^{-m};a,b,c;q) = \frac{(bq;q)_m}{(aq;q)_m}(-a)^m q^{m(m+1)/2} D_n(\mu(m);b,a,ab/c|q).$$

Formula (8.5) directly leads to the orthogonality relation for the polynomials $D_n(\mu(m)) \equiv D_n(\mu(m);a,b,c|q)$:

$$\sum_{m=0}^\infty \frac{(1-abq^{2m+1})(aq,abq,abq/c;q)_m}{(1-abq)(bq,cq,q;q)_m} (-c/a)^m q^{m(m-1)/2} D_n(\mu(m)) D_{n'}(\mu(m))$$

18

$$= \frac{(abq^2, c/a; q)_\infty}{(bq, cq; q)_\infty} \frac{(aq/c, q; q)_n}{(aq, abq/c; q)_n q^n} \delta_{nn'}. \tag{8.9}$$

From (8.6) one obtains the orthogonality relation for the polynomials $D_n(\mu(m); b, a, ab/c|q)$ (which follows also from the relation (8.9) by interchanging a and b and replacing c by ab/c).

We call the polynomials $D_n(\mu(m); a, b, c|q)$ *dual big q-Jacobi polynomials*. It is natural to ask whether they can be identified with some known and thoroughly studied set of polynomials. The answer is: they can be obtained from the q-Racah polynomials $R_n(\mu(x); a, b, c, d|q)$ of Askey and Wilson [29] by setting $a = q^{-N-1}$ and sending $N \to \infty$, that is,

$$D_n(\mu(x); a, b, c|q) = \lim_{N \to \infty} R_n(\mu(x); q^{-N-1}, a/c, a, b|q). \tag{8.10}$$

Observe that the orthogonality relation (8.9) can be also derived from formula (4.16) in [30]. But the derivation of this formula (4.16) is rather complicated.

It is worth noting here that in the limit as $c \to 0$ the dual big q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ coincide with the dual little q-Jacobi polynomials $d_n(\mu(x); b, a|q)$, defined in section 5. The dual little q-Jacobi polynomials $d_n(\mu(x); a, b|q)$ reduce, in turn, to the Al-Salam–Carlitz II polynomials $V_n^{(a)}(s; q)$ on the q-linear lattice $s = q^{-x}$ (see [19], p. 114) in the case when the parameter b vanishes, that is,

$$d_n(\mu(x); a, 0|q) = {}_2\phi_0(q^{-n}, q^{-x}; -; q, q^n/a) = (-a)^{-n} q^{n(n-1)/2} V_n^{(a)}(q^{-x}; q). \tag{8.11}$$

This means that we have now a complete chain of reductions

$$R_n(\mu(x); a, b, c, d|q) \xrightarrow[a \to \infty]{} D_n(\mu(x); b, c, d|q) \xrightarrow[d \to 0]{} d_n(\mu(x); c, b|q) \xrightarrow[b=0]{} V_n^{(c)}(q^{-x}; q)$$

from the four-parameter family of q-Racah polynomials, which occupy the upper level in the Askey-scheme of basic hypergeometric polynomials (see [19], p. 62), down to the one-parameter set of Al-Salam–Carlitz II polynomials from the second level in the same scheme. So, the dual big and dual little q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ and $d_n(\mu(x); a, b|q)$ should occupy the fourth and third level in the Askey-scheme, respectively.

The recurrence relations for the polynomials $D_n(\mu(m) \equiv D_n(\mu(m); a, b, c|q)$ are obtained from the q-difference equation (6.6). It has the form

$$(q^{-m} - 1)(1 - abq^{m+1})D_n(\mu(m)) = A_n D_{n+1}(\mu(m)) - (A_n + C_n)D_n(\mu(m)) + C_n D_{n-1}(\mu(m)),$$

where

$$A_n = (1 - aq^{n+1})(1 - cq^{n+1}), \quad C_n = aq(1 - q^n)(b - cq^n).$$

The relation (8.7) leads to the equality

$$\sum_{m=0}^{\infty} (-1)^m \frac{(1 - abq^{2m+1})(abq; q)_m}{(1 - abq)(q; q)_m} q^{m(m-1)/2} D_n(\mu(m); a, b, c|q) D_{n'}(\mu(m); b, a, abq/c|q) = 0. \tag{8.12}$$

We give an alternative proof of this result in Appendix.

Note that from the expression (8.8) for the dual big q-Jacobi polynomials $D_n(\mu(m); a, b, c|q)$ it follows that they possess the symmetry property

$$D_n(\mu(m); a, b, c|q) = D_n(\mu(m); ab/c, c, b|q). \tag{8.13}$$

The set of functions (8.1) and (8.2) form an orthogonal basis in the Hilbert space l^2 of functions, defined on the set of points $m = 0, 1, 2, \cdots$, with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \rho(m)\, f_1(m)\, \overline{f_2(m)},$$

where $\rho(m)$ is the same as in formulas (8.5)–(8.7). Consequent from this fact, one can deduce (in the same way as in the case of dual little q-Jacobi polynomials) that *the dual big q-Jacobi polynomials $D_n(\mu(m); a, b, c|q)$ correspond to indeterminate moment problem and the orthogonality measure for them, given by formula (8.9), is not extremal.*

It is difficult to find extremal measures. As far as we know, explicit forms of extremal measures have been constructed only for the q-Hermite polynomials with $q > 1$ (see [31]).

9. Generating functions

Generating functions are known to be of great importance in the theory of orthogonal polynomials (see, for example, [16]). For the sake of completeness, we briefly discuss in this section some instances of linear generating functions for the dual q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ and $d_n(\mu(x); a, b|q)$. To start with, let us consider a generating-function formula

$$\sum_{n=0}^{\infty} \frac{(aq; q)_n}{(q; q)_n}\, t^n D_n(\mu(x); a, b, c|q) = \frac{(aqt; q)_{\infty}}{(t; q)_{\infty}}\, {}_2\phi_2 \left(\begin{array}{c} q^{-x},\ abq^{x+1} \\ abq/c,\ aqt \end{array} \middle|\, q,\ aqt/c \right), \tag{9.1}$$

where $|t| < 1$ and, as before, $\mu(x) = q^{-x} + abq^{x+1}$. To verify (9.1), insert the explicit form (8.8) of the dual big q-Jacobi polynomials

$$D_n(\mu(x); a, b, c|q) = \sum_{k=0}^{n} \frac{(q^{-x}, abq^{x+1}, q^{-n}; q)_k}{(aq, abq/c, q; q)_k} \left(\frac{aq^{n+1}}{c} \right)^k$$

into the left side of (9.1) and interchange the order of summation. The subsequent use of the relations

$$(a; q)_{m+k} = (a; q)_m (aq^m; q)_k = (a; q)_k (aq^k; q)_m,$$

$$(q^{-m-k}; q)_k = (-1)^k q^{-mk - k(k+1)/2} (q^{m+1}; q)_k$$

(see [15], Appendix I) simplifies the inner sum and enables one to evaluate it by the q-binomial formula (4.9). This gives the quotient of two infinite products in front of ${}_2\phi_2$ on the right side of (9.1), times $(aqt; q)_k^{-1}$. The remaining sum over k yields ${}_2\phi_2$ series itself.

As a consistency check, one may also obtain (9.1) directly from the generating function for the q-Racah polynomials $R_n(\mu(x); \alpha, \beta, \gamma, \delta|q)$ (see formula (3.2.13) in [19]) by setting $\alpha = q^{-N-1}$ and sending $N \to \infty$. This results in the relation

$$\sum_{n=0}^{\infty} \frac{(aq; q)_n}{(q; q)_n}\, t^n D_n(\mu(x); a, b, c|q) = \frac{(aq^{x+1}t; q)_{\infty}}{(t; q)_{\infty}}\, {}_2\phi_1 \left(\begin{array}{c} q^{-x},\ c^{-1}q^{-x} \\ abq/c \end{array} \middle|\, q,\ atq^{x+1} \right). \tag{9.2}$$

The left side of (9.2) depends on the variable x by dint of the combination $\mu(x) = q^{-x} + abq^{x+1}$. Off hand, it is not evident that the right side of (9.2) is also a function of the lattice $\mu(x)$.

Nevertheless, this is the case. Moreover, the right sides of (9.1) and (9.2) are equivalent: this fact is known in the theory of special functions as Jackson's transformation

$$_2\phi_1(a,b;\ c;\ q,z) = \frac{(az;q)_\infty}{(z;q)_\infty}\,_2\phi_2(a,c/b;\ c,az;\ q,bz)$$

(see, for example, [15]).

The symmetry property (8.13) of the dual big q-Jacobi polynomials $D_n(\mu(x);a,b,c|q)$, combined with (9.1), generates another relation

$$\sum_{n=0}^{\infty}\frac{(abq/c;q)_n}{(q;q)_n}t^n D_n(\mu(x);a,b,c|q) = \frac{(abqt/c;q)_\infty}{(t;q)_\infty}\,_2\phi_2\left(\begin{array}{c}q^{-x},\ abq^{x+1}\\ aq,\ abqt/c\end{array}\bigg|\,q,\ aqt/c\right)$$

$$= \frac{(abtq^{x+1}/c;q)_\infty}{(t;q)_\infty}\,_2\phi_1\left(\begin{array}{c}q^{-x},\ b^{-1}q^{-x}\\ aq\end{array}\bigg|\,q,\ abtq^{x+1}/c\right). \tag{9.3}$$

Similarly, a generating function for the dual little q-Jacobi polynomials has the form

$$\sum_{n=0}^{\infty}\frac{(bq;q)_n}{(q;q)_n}(at)^n d_n(\mu(x);a,b|q) = \frac{(tq^{-x},abtq^{x+1};q)_\infty}{(at,t;q)_\infty}. \tag{9.4}$$

One can verify (9.4) directly by inserting the explicit form (5.4) of $d_n(\mu(x);a,b|q)$ into the left side of (9.4) and repeating the same steps as in the case of deriving (9.1). This will lead to the expression

$$\frac{(abqt;q)_\infty}{(at;q)_\infty}\,_2\phi_1(q^{-x},\ abq^{x+1};\ abqt;\ q,t)$$

and it remains only to employ Heine's summation formula (1.5.1) from [15]. After a simple rescaling of the parameters the generating function (9.4) coincides with that, obtained earlier in [24].

The simplest way of obtaining (9.4) is to send $c \to 0$ in both sides of (9.2): the $_2\phi_1$ series on the right side of (9.2) reduces to $_1\phi_0(q^{-x};-;\ q,t/a)$, which is evaluated by the q-binomial formula (4.9).

Finally, when the parameter b vanishes, (9.4) reduces to the known generating function

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}t^n V_n^{(a)}(q^{-x};q) = \frac{(tq^{-x};q)_\infty}{(at,t;q)_\infty}$$

for the Al-Salam–Carlitz II polynomials (see (3.25.11) in [19]).

10. Concluding remarks

To summarize, we have attempted to carry over, by using the discrete series representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$, an idea of the duality of polynomials, orthogonal on a finite set of points, to the case of big and little q-Jacobi polynomials.

In fact, we used this idea in [6] and [8] to show that Al-Salam–Carlitz II polynomials are dual with respect to little q-Laguerre polynomials and q-Meixner polynomials are dual to big q-Laguerre polynomials, respectively.

This approach may be further explored and applied to other q-polynomials families. In particular, we know that it is possible to show, by using certain irreducible representations

of the quantum algebra $U_q(\mathrm{su}_{1,1})$ (which are not $*$-representations of $U_q(\mathrm{su}_{1,1})$), that polynomials, dual to q-Charlier polynomials (see formula (3.23.1) in [19]), are Al-Salam–Carlitz polynomials I (see formula (3.24.1) in [19]) and polynomials, dual to the alternative q-Charlier polynomials

$$K_n(x; a; q) := {}_2\phi_1(q^{-n}, -aq^n;\ 0;\ q; qx), \quad n = 0, 1, 2, \cdots,$$

(see formula (3.22.1) in [19]), are the polynomials

$$d_m(\mu(n); a; q) = {}_3\phi_0(q^{-n}, -aq^n, q^{-m};\ -;\ q, -q^m/a), \quad m = 0, 1, 2, \cdots,$$

where $\mu(n) = q^{-n} - aq^n$. The orthogonality relation for them has the form

$$\sum_{n=0}^{\infty} \frac{(-a; q)_n (1 + aq^{2n}) a^n}{(q; q)_n}\, q^{(3n-1)n/2} d_m(\mu(n)) d_{m'}(\mu(n)) = \frac{(-a; q)_\infty (q; q)_m}{a^m q^{m(m+1)/2}}\, \delta_{mm'}, \quad a > 0.$$

Proofs of these statements will be given in a separate publication.

Acknowledgments

Appendix

In this appendix we prove the summation formula

$$\sum_{n=0}^{\infty} \frac{(abq, bq; q)_n}{(aq, q; q)_n} \frac{1 - abq^{2n+1}}{1 - abq} a^n q^{n^2} = \frac{(abq^2; q)_\infty}{(aq; q)_\infty}. \tag{A.1}$$

First of all, observe that when $b = 0$ this relation reduces to

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(aq, q; q)_n} = \frac{1}{(aq; q)_\infty},$$

which is a well-known limiting form of Jacobi's triple product identity (see [15], formula (1.6.3)).

One can employ an easily verified relation

$$\frac{(aq, -aq; q)_n}{(a, -a; q)_n} = \frac{1 - a^2 q^{2n}}{1 - a^2} \tag{A.2}$$

in order to express the infinite sum in (A.1) in terms of a very-well-poised ${}_4\phi_5$ basic hypergeometric series. This results in

$$\sum_{n=0}^{\infty} \frac{(abq, bq; q)_n}{(aq, q; q)_n} \frac{1 - abq^{2n+1}}{1 - abq} a^n q^{n^2} = {}_4\phi_5 \left(\begin{array}{c} abq,\ bq,\ q\sqrt{abq},\ -q\sqrt{abq} \\ aq,\ \sqrt{abq},\ -\sqrt{abq},\ 0,\ 0 \end{array} \middle|\ q, aq \right). \tag{A.3}$$

The next step is to utilize a limiting case of Jackson's sum of a terminating very-well-poised balanced $_8\phi_7$ series,

$$_6\phi_5\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d \end{matrix}\ \middle|\ q,\ \frac{aq}{bcd}\right) = \frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}, \qquad (A.4)$$

which represents a q-analogue of Dougall's formula for a very-well-poised 2-balanced $_7F_6$ series. When the parameters c and d tend to infinity, from (A.4) it follows that

$$_4\phi_5\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ 0,\ 0 \end{matrix}\ \middle|\ q,\ \frac{aq}{b}\right) = \frac{(aq;q)_\infty}{(aq/b;q)_\infty}. \qquad (A.5)$$

To verify this, one needs only to use the limit relation

$$\lim_{c,d\to\infty}(c,d;q)_n\left(\frac{aq}{bcd}\right)^n = q^{n(n-1)}\left(\frac{aq}{b}\right)^n.$$

With the substitutions $a \to abq$ and $b \to bq$ in (A.5), one recovers the desired identity (A.1). Similarly, when $d \to \infty$ we derive from (A.4) the identities

$$\sum_{n=0}^\infty \frac{(1-abq^{2n+1})(aq,abq/c,abq;q)_n}{(1-abq)(bq,cq,q;q)_n(-a/c)^n}q^{n(n-1)/2} = \frac{(abq^2,c/a;q)_\infty}{(bq,cq;q)_\infty}, \qquad (A.6)$$

$$\sum_{n=0}^\infty \frac{(1-abq^{2n+1})(abq,bq,cq;q)_n}{(1-abq)(aq,abq/c,q;q)_n(-c/a)^n}q^{n(n-1)/2} = \frac{(abq^2,a/c;q)_\infty}{(aq,abq/c;q)_\infty}. \qquad (A.7)$$

They have been used in section 7.

We conclude this appendix with the following remark. There is another proof of the identity (8.12), based on vital use of the same summation formula (A.4). Actually, a relation may be derived, which is somewhat more general than (8.12). Indeed, consider the function

$$\eta_k(a;q) := \sum_{n=0}^\infty (-1)^n q^{n(n-1)/2}\frac{1-aq^{2n+1}}{1-aq}\frac{(aq;q)_n}{(q;q)_n}\mu^k(n;a) \qquad (A.8)$$

for arbitrary nonnegative integers k, where the q-quadratic lattice $\mu(n;a)$ is defined as before:

$$\mu(n;a) := q^{-n} + aq^{n+1}. \qquad (A.9)$$

We argue that all $\eta_k(a;q) = 0$, $k = 0,1,2,\cdots$. To verify that, begin with the case when $k = 0$ and employ relation (A.2) to show that

$$\eta_0(a;q) = {}_3\phi_3\left(\begin{matrix} q\sqrt{aq},\ -q\sqrt{aq},\ aq \\ \sqrt{aq},\ -\sqrt{aq},\ 0 \end{matrix}\ \middle|\ q,1\right).$$

The summation formula (A.4) in the limit as $d \to \infty$ takes the form

$$_5\phi_5\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ 0 \end{matrix}\ \middle|\ q,\ \frac{aq}{bc}\right) = \frac{(aq,aq/bc;q)_\infty}{(aq/b,aq/c;q)_\infty}. \qquad (A.10)$$

In the particular case when $bc = aq$ this sum reduces to

$$_3\phi_3\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a} \\ 0,\ \sqrt{a},\ -\sqrt{a} \end{matrix}\ \middle|\ q,1\right) = \frac{(aq,1;q)_\infty}{(b,c;q)_\infty} = 0,$$

23

since $(z;q)_\infty = 0$ for $z = 1$. Consequently, the function $\eta_0(a;q)$ does vanish.

For $k = 1, 2, 3, \cdots$, one can proceed inductively. Employ the relation $q\mu(n+1;a) = \mu(n;q^2a)$ to show that

$$\eta_{k+1}(a;q) = (1+aq)\eta_k(a;q) - q^{-k-1}(1-aq^2)(1-aq^3)\eta_k(aq^2;q).$$

So, one obtains that indeed all $\eta_k(a;q)$, $k = 0, 1, 2, \cdots$, vanish. The identity (8.12) is now an easy consequence of this statement if one takes into account that a product of the two polynomials $D_n(\mu(m);a,b,c|q)$ and $D_{n'}(\mu(m);b,a,abq/c|q)$ in (8.12) is some polynomial in $\mu(m)$ of degree $n + n'$. This completes the proof of (8.12), which is independent of the one, given in section 8.

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